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TAIL FIELD REPRESENTATIONS AND THE ZERO-TWO LAW*

BY

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ABSTRACT

The zero-two law was proved for a positive L_1-contraction T by Ornstein and Sucheston, and gives a condition which implies $T^n f - T^{n+1} f \to 0$ for all f. Extensions of this result to the case of a positive L_p-contraction, $1 \leq p < \infty$, have been obtained by several authors. In the present paper we prove a theorem which is related to work of Wittmann.

We will say that a positive contraction T contains a circle of length m if there is a nonzero function f such that the iterated values $f, Tf, \ldots, T^{m-1}f$ have disjoint support, while $T^m f = f$. Similarly, a contraction T contains a line if for every m there is a nonzero function f (which may depend on m) such that $f, Tf, \ldots, T^{m-1}f$ have disjoint support. Approximate forms of these conditions are defined, which are referred to as asymptotic circles and lines, respectively. We show (Theorem 3) that if the conclusion $T^n f - T^{n+1} f \to 0$ of the zero-two law does not hold for all f in L_p, then either T contains an asymptotic circle or T contains an asymptotic line. The point of this result is that any condition on T which excludes circles and lines must then imply the conclusion of the zero-two law.

Theorem 3 is proved by means of the representation of a positive L_p-contraction in terms of an L_p-isometry. Asymptotic circles and lines for

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T correspond to exact circles and lines for the isometry on tail-measurable functions, and exact circles and lines for the isometry are obtained using the Rohlin tower construction for point transformations.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a measure space, and let p be real, $1 \leq p \leq \infty$. For any bounded operator T from $L_p(X, \mathcal{F}, \mu)$ to $L_p(X, \mathcal{F}, \mu)$ such that $Tf \geq 0$ whenever $f \geq 0$, we will say that T is positive. We write $L_p^+(X, \mathcal{F}, \mu)$ to denote the nonnegative elements of $L_p(X, \mathcal{F}, \mu)$. If $\|T\| \leq 1$ we will say that T is a contraction. In 1970 Ornstein and Sucheston proved the zero-two law for a positive L_1-contraction. Their basic result, given as Theorem 1.1 in [12], was the following.

THEOREM 1 (Ornstein and Sucheston): *Let T be a positive linear contraction on L_1. Assume there is some $\varepsilon > 0$ such that for every $f \in L_1^+$ we have*

$$\lim_{n\to\infty} \|T^n f - T^{n+1} f\|_1 \leq 2(1-\varepsilon)\|f\|_1. \tag{1}$$

Then for every $f \in L_1$ we have

$$\lim_{n\to\infty} \|T^n f - T^{n+1} f\|_1 = 0. \tag{2}$$

Several extensions of this theorem are known, for example [8], [9], [7], [18], [10], [14], [16], [17], [19]. In particular, extensions to the case of operators on L_p have been obtained. The main result of the present paper, stated below as Theorem 3, is another extension of the zero-two law in L_p, $1 \leq p < \infty$. Before stating Theorem 3 we will review some related earlier work.

The first zero-two law in L_p for $p > 1$ was obtained by Zaharopol [18], for $1 \leq p < \infty$, $p \neq 2$. Katznelson and Tzafriri [10] used a different method and were able to prove the same result for contractions on function spaces in a class which includes L_p for all p with $1 \leq p < \infty$. These results gave convergence in uniform norm for $T^n - T^{n+1}$, and in the hypotheses assumed asymptotic bounds on the *linear modulus* of $T^n - T^{n+1}$, which we will write as $|T^n - T^{n+1}|$. The linear modulus $|T^n - T^{n+1}|$ is, in general, a larger operator than $T^n - T^{n+1}$. Later, in [17], Theorems 1.1 and 2.2, uniform and strong operator convergence results were proved in which both the hypotheses and the conclusions were based on $|T^n - T^{n+1}|$. In [19], Theorem E, uniform operator convergence for $|T^n - T^{n+1}|$ was obtained in the setting of *Banach lattices* (cf. [19], [15]).

Another criterion for strong operator convergence in L_p was given in [16], and, unlike the results just mentioned, is not formulated in terms of the linear modulus $|T^n - T^{n+1}|$.

THEOREM 2 (Wittmann): *Let T be a positive linear contraction on L_p. Let α_p be defined for $1 \le p < \infty$ by*

$$\alpha_p = \sup_{0 \le x \le y} \left(\frac{(1+x)^p + (1+y)^p + (y-x)^p}{1 + x^p + y^p} \right)^{1/p}. \tag{3}$$

The following conditions are equivalent:

(i)

$$\lim_{n\to\infty} \|T^n f - T^{n+1} f\|_p = 0 \quad \text{for every } f \in L_p. \tag{4}$$

(ii) *There exists some $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} \|T^n f - T^{n+1} f\|_p \le (1-\varepsilon) 2^{1/p} \|f\|_p \quad \text{for every } f \in L_p^+. \tag{5}$$

(iii) *There exists some $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} \|T^n f - T^{n+1} f\|_p \le (1-\varepsilon) \alpha_p \|f\|_p \quad \text{for every } f \in L_p. \tag{6}$$

(iv) *There exists some $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} \|(T^n f - T^{n+1} f)^+\|_p \le (1-\varepsilon) \|f\|_p \quad \text{for every } f \in L_p^+. \tag{7}$$

Theorem 2 is not implied by the earlier results mentioned, since the conditions of Theorem 2 are satisfied whenever $T^n - T^{n+1} \to 0$ in the strong operator topology, and easy examples exist of positive contractions T such that $T^n - T^{n+1}$ converges to 0 in the strong operator topology but the linear modulus $|T^n - T^{n+1}|$ does not converge to 0 in any sense.

The quantity α_p in Theorem 2 is related to certain particularly simple L_p-contractions. For each $m = 1, 2, 3, \ldots$, let τ_m be defined by τ_m: $\{1, \ldots, m\} \to \{1, \ldots, m\}$, $\tau_m(i) = i+1$ for $i = 1, \ldots, m-1$, $\tau_m(m) = 1$. Let μ_m be the counting measure on $\{1, \ldots, m\}$. Let T_m be the operator defined by $T_m f = f \circ \tau_m$. T_m is an isometry on $L_p(\mu_m)$ for every p. We will refer to T_m as the circle shift isometry of length m. Similarly, let τ_∞ be the shift on the integers defined by τ_∞: $\mathbb{Z} \to \mathbb{Z}$, $\tau_\infty(i) = i + 1$ for all $i \in \mathbb{Z}$, let μ_∞ denote the counting measure on $\mathbb{Z}$, and let $T_\infty f = f \circ \tau$, so that T_∞ is an isometry on $L_p(\mu_\infty)$ for every p. We will refer to T_∞ as the infinite line shift isometry.

Let $\alpha_p(m) = \|I - T_m\|$, where $I - T_m$ is considered as an operator on $L_p(\mu_m)$, $1 \le p \le \infty$, and define $\alpha_p(\infty)$ similarly. Clearly, $\alpha_p(m) \le 2$ for all m and all p. By considering the function $f(i) = (-1)^i$, it is easy to see that $\alpha_p(m) = 2$ for m even and for $m = \infty$, and (cf. [16]) that $\alpha_p(3)$ is equal to the quantity α_p defined in Theorem 2. Using the Riesz Convexity Theorem much as in [16], we see readily that $\alpha_p(m) \ge \alpha_p(3)$ for all $m \ge 2$. Thus we conclude that α_p is the smallest value of $\|I - T\|$ for any T which is a circle or an infinite line shift isometry.

Using the remarks just made, it is almost immediate that each of the statements in Theorem 2 is false if the contraction T in the theorem is a circle shift isometry or an infinite line shift isometry. This fact, together with an example given in [10], motivates the next definition.

Definition 1: Let T be a positive linear contraction on L_p, for some p with $1 \le p < \infty$. For any positive integer m, we will say that the T contains an asymptotic circle of length m if for every $\varepsilon > 0$ there exists $f \in L_p^+$ with $\lim_{n\to\infty} \|T^n f\|_p \ge 1$ such that for all sufficiently large n we have

$$\|T^{n+i} f \wedge T^{n+j} f\|_p < \varepsilon \quad \text{for } 0 \le i < j < m, \tag{8}$$

and

$$\|T^{n+m} f - T^n f\|_p < \varepsilon. \tag{9}$$

Similarly, for any positive integer m, we will say that T contains an asymptotic line segment of length m if for every $\varepsilon > 0$ there exists $f \in L_p^+$ with $\lim_{n\to\infty} \|T^n f\|_p \ge 1$ such that for all sufficiently large n, condition (8) holds.

The point of this definition is given in the next theorem.

THEOREM 3: *Let T be a positive linear contraction on L_p, for some p with $1 \le p < \infty$. Consider the following two statements.*

(a) *T contains an asymptotic circle of length $m \ge 2$.*

(b) *T contains an "asymptotic infinite line", that is, T contains asymptotic line segments of length m for every positive integer m.*

Suppose neither of these statements (a) *and* (b) *is true. Then for every $f \in L_p$ we have*

$$\lim_{n\to\infty} \|T^n f - T^{n+1} f\|_p = 0. \tag{10}$$

Theorem 3 easily implies Theorem 2, since conditions (a) and (b) of Theorem 3 are incompatible with each of the conditions (i)–(iv) of Theorem 2, by the same arguments which work for the exact circle and line shift isometries.

In the proof of Theorem 3 we will consider the properties of an associated L_p-isometry and its tail σ-algebra. This approach is similar to that used by Derriennic [7], [13] to prove the zero-two law for L_1. The general form of the associated L_p-isometry is given in [4], [6] and [3], and the technique of representing a positive L_p-contraction via an L_p-isometry has been to obtain other convergence theorems (cf. [1], [3]). In the present situation the action of the L_p-isometry on L_p-functions which are tail-measurable (cf. Section 3.3) captures some of the asymptotic behavior of T^n, while being easier to analyze. In particular, we can relate the asymptotic circle and line cases for T to corresponding *exact* circle and line cases which are defined later for the isometry. These exact circle and line cases are obtained in a very simple way by constructing Rohlin towers for the nonsingular point transformation which induces the L_p-isometry (Lemma 7).

It is natural to consider whether an analog of Theorem 3 can be formulated which will help in analyzing the convergence of the modulus operator $|T^n - T^{n+1}|$, but we have not found such a result.

2. The dilation case

We will shortly give the proof of Theorem 3 in a special case, which we now describe. We will refer to the situation defined in the present section as the dilation case. It is known ([4], [6], [3]) that a rather general positive L_p-contraction T is equivalent via an L_p-isometry to an operator which satisfies the assumptions of the dilation case. For this reason, as we will show later, the general proof of Theorem 3 is a consequence of the proof in the dilation case.

Let (Ω, Σ, γ) be a probability space, and let $\mathcal{F}$ be a sub-σ-algebra of Σ. Let $\tau\colon \Omega \to \Omega$ be a measurable and invertible point transformation on (Ω, Σ, γ), meaning that τ is one-to-one and onto and both τ and τ^{-1} are measurable from Σ to Σ. We assume that τ and τ^{-1} are nonsingular, so that $\gamma(\tau^{-1}(A)) = 0$ if and only if $\gamma(A) = 0$, for all $A \in \Sigma$.

We will define the change of measure factor ρ associated with τ as the Radon–Nikodym derivative $d\gamma\tau^{-1}/d\gamma$. Of course, for any nonnegative Σ-measurable function f we have

$$\int f \circ \tau d\gamma = \int \rho f d\gamma. \tag{11}$$

Since τ^{-1} is assumed to be nonsingular we have $\rho > 0$, γ-almost everywhere.

For any finite Σ-measurable function f on Ω we define

$$Wf = \rho(f \circ \tau^{-1}). \tag{12}$$

By (11), W is an isometry on $L_1(\Omega, \Sigma, \gamma)$. We note that $\rho = W\mathbf{1}$. The operator W is almost multiplicative, since

$$W(fg) = (f \circ \tau^{-1})Wg. \tag{13}$$

We also define $W_p f = \rho^{1/p}(f \circ \tau^{-1})$, so that $Wf^p = (W_p f)^p$ for $f \geq 0$. We see at once that W_p is an isometry on $L_p(\Omega, \Sigma, \gamma)$. We will say that W_p is the L_p-isometry associated with the nonsingular point transformation τ.

For any integer n, let $\mathcal{F}_n = \tau^{-n}(\mathcal{F})$, let $\mathcal{G}_n$ be the σ-algebra generated by the union of $\mathcal{F}_m$, $m \leq n$, and let $\mathcal{H}_n$ be the σ-algebra generated by the union of $\mathcal{F}_m$, $m \geq n$.

We assume that the following "Markov" properties hold for each nonnegative integer n:

$$E[Wf|\mathcal{H}_n] = E[Wf|\mathcal{F}_n] \tag{14}$$

for any $f \in L_1(\Omega, \mathcal{G}_n, \gamma)$,

$$E[Wf|\mathcal{G}_n] = E[Wf|\mathcal{F}_n] \tag{15}$$

for any $f \in L_1(\Omega, \mathcal{H}_n, \gamma)$. Here, as in what follows, we use E to denote the operation of conditional expectation using the probability measure γ. It is an easy standard fact that either of (14) or (15) implies the other.

We assume that we wish to study a positive L_p-contraction T from $L_p(\Omega, \mathcal{F}, \gamma)$ to itself, with the following key property:

$$Tf \leq E[W_p f|\mathcal{H}_0] \tag{16}$$

for any $f \in L_p^+(\Omega, \mathcal{F}, \gamma)$.

The operator W_p is sometimes referred to as a dilation of T, especially when equality holds in (16). It is for this reason that we refer to the situation described in the present section as the dilation case.

Under the assumptions of the dilation case we will be able to carry out the proof of Theorem 3, using τ as the main tool. Moreover, as mentioned earlier, we can reduce the general case to this situation.

3. Proof of Theorem 3 in the dilation case

3.1 PROPERTIES OF τ. We begin by noting the relation between τ and the operation of taking conditional expectation. To make the notation a little more readable, for any sub-σ-algebra $\mathcal{K}$ of Σ we will sometimes write $E[f|\mathcal{K}]$ as $E_{\mathcal{K}}f$. The next lemma applies to any nonsingular point transformation, and makes no use of (14) or (14).

LEMMA 1: *Let $\mathcal{K}$ be any sub-σ-algebra of Σ. Let τ be any measurable and invertible point transformation which is also nonsingular. Then*

$$E_{\tau(\mathcal{K})}W = E_{\tau(\mathcal{K})}WE_{\mathcal{K}}. \tag{17}$$

Proof: Let $f \in L_1(\Omega, \Sigma, \gamma)$, and let $h = E_{\mathcal{K}}f$. For any $A \in \mathcal{K}$, since $\mathbf{1}_A \circ \tau^{-1} = \mathbf{1}_{\tau(A)}$, by (13) and the isometry property of W we have

$$\int_{\tau(A)} E_{\tau(\mathcal{K})}Wf\,d\gamma = \int_{\tau(A)} Wf\,d\gamma = \int W\mathbf{1}_A f\,d\gamma = \int_A f\,d\gamma. \tag{18}$$

The same equation of course holds with f replaced by h. Thus

$$\int_A f\,d\gamma = \int_A h\,d\gamma = \int_{\tau(A)} E_{\tau(\mathcal{K})}Wh\,d\gamma, \tag{19}$$

and the lemma follows. ∎

By Lemma 1 we see easily that for any integer k

$$E_{\mathcal{H}_k}W = E_{\mathcal{H}_k}WE_{\mathcal{H}_{k+1}} \tag{20}$$

and hence that for each nonnegative integer n we have

$$E_{\mathcal{H}_k}W^n = E_{\mathcal{H}_k}W^nE_{\mathcal{H}_{k+n}}. \tag{21}$$

Since $E_{\mathcal{H}_0}E_{\mathcal{H}_{-1}} = E_{\mathcal{H}_0}$, it follows from the case $k = -1$ of (20) that

$$E_{\mathcal{H}_0}W = E_{\mathcal{H}_0}WE_{\mathcal{H}_0}. \tag{22}$$

Let $f \in L_1(\Omega, \mathcal{H}_{k+n}, \gamma)$. Then

$$E_{\mathcal{H}_k}W^nf = (E_{\mathcal{H}_k}W^n\mathbf{1})f \circ \tau^{-n}, \tag{23}$$

since $W^nf = (W^n\mathbf{1})f \circ \tau^{-n}$ and $f \circ \tau^{-n}$ is $\mathcal{H}_k$-measurable. It follows easily from (23) that the restriction of the map $E_{\mathcal{H}_k}W^n$ to $L_1(\Omega, \mathcal{H}_{k+n}, \gamma)$ is an L_1-isometry, and so the map from $L_p(\Omega, \mathcal{H}_{k+n}, \gamma)$ to $L_p(\Omega, \mathcal{H}_k, \gamma)$ given by

$$g \mapsto (E_{\mathcal{H}_k}W^n\mathbf{1})^{1/p}g \circ \tau^{-n} \tag{24}$$

is an L_p-isometry.

We also have for any nonnegative $\mathcal{H}_k$-measurable function f that

$$\int f \circ \tau^n d\gamma = \int (E_{\mathcal{H}_k} W^n \mathbf{1}) f d\gamma. \tag{25}$$

Indeed, when f is bounded, we can write $f = g \circ \tau^{-n}$, for some bounded g, and (25) follows from the L_1-isometry property just stated. We may then apply the Monotone Convergence Theorem.

It follows readily from (25) that the isometry defined by (24) is in fact onto from $L_p(\Omega, \mathcal{F}_{k+n}, \gamma)$ to $L_p(\Omega, \mathcal{F}_k, \gamma)$. When $k = 0$ we will denote the inverse isometry from $L_p(\Omega, \mathcal{F}_0, \gamma)$ onto $L_p(\Omega, \mathcal{F}_n, \gamma)$ by Λ_n. Clearly Λ_n is given by

$$\Lambda_n g = \left(\frac{g}{(E_{\mathcal{H}_0} W^n \mathbf{1})^{1/p}} \right) \circ \tau^n. \tag{26}$$

3.2 Limits of Moving Iterates.

3.2.1 Motivation. We define the positive $L_1(\Omega, \mathcal{F}_0, \gamma)$-contraction U, as follows:

$$Uf \equiv E_{\mathcal{F}_0} W f. \tag{27}$$

It is easy to show that $(Tf)^p \leq U f^p$ for all $f \in L_p^+(\Omega, \mathcal{F}_0, \gamma)$, and also, using (22), that $E_{\mathcal{H}_0} W^n \mathbf{1} = U^n \mathbf{1}$. Though we will not make use of U later, this operator gives some insight into the long-term behavior of the ratio appearing in (26).

The convergence of $T^n f - T^{n+1} f$ would be trivial if we could show that $T^n f$ converges to a limit, but unfortunately $T^n f$ need not converge. However, we can adapt the following old observation from Markov chain theory. Consider a Markov chain (ξ_n), and suppose that ξ_n has density f_n with respect to P^{f_0} and density g_n with respect to P^{g_0}, where f_0 and g_0 are two possible initial densities. If we assume that $\{f_0 > 0\} \subset \{g_0 > 0\}$ (and hence that $\{f_n > 0\} \subset \{g_n > 0\}$), it is easy to show that $(f_n/g_n) \circ \xi_n$ forms a backwards martingale with respect to P^{g_0}, and thus converges very nicely, even though f_n may not converge at all. Intuitively, we obtain convergence of the ratio by observing the values of f_n using an observer that moves with the process.

In our present situation we will use a slightly more complicated ratio involving both T and U, but the basic type of convergence is the same. Of course, the relevance of the limit we obtain must be shown later.

3.2.2 Proving convergence.

Lemma 2: *Let ℓ be a nonnegative integer. Let $f \in L_p^+(\Omega, \mathcal{F}_0, \gamma)$. Then the sequence $((\Lambda_n T^{n-\ell} f)^p, \mathcal{H}_n)_{n=\ell,\ell+1,\ldots}$ is a backward submartingale.*

Proof: Let $u_n = (\Lambda_n T^{n-\ell} f)^p$. Using (23),

$$E_{\mathcal{H}_0} W^{n+1} u_{n+1} = (T^{n+1-\ell} f)^p \tag{28}$$

$$\leq (E_{\mathcal{H}_0} W_p T^{n-\ell} f)^p \tag{29}$$

$$\leq E_{\mathcal{H}_0} (W_p T^{n-\ell} f)^p \tag{30}$$

$$= E_{\mathcal{H}_0} W (T^{n-\ell} f)^p \tag{31}$$

$$= E_{\mathcal{H}_0} W E_{\mathcal{H}_0} W^n u_n \tag{32}$$

$$= E_{\mathcal{H}_0} W^{n+1} u_n \tag{33}$$

$$= E_{\mathcal{H}_0} W^{n+1} E_{\mathcal{H}_{n+1}} u_n. \tag{34}$$

Since the restriction of $E_{\mathcal{H}_0} W^{n+1}$ to $L_1(\Omega, \mathcal{H}_{n+1}, \gamma)$ is an order-preserving map, by (23), the lemma follows. ∎

As a consequence of Lemma 2, for any $f \in L_p^+(\Omega, \mathcal{F}_0, \gamma)$ the sequence $(\Lambda_n T^{n-\ell} f)^p$ converges both a.e. and in L_1-norm. Hence the sequence $\Lambda_n T^{n-\ell} f$ converges both a.e. and in L_p-norm. By linearity this conclusion then holds for all $f \in L_p(\Omega, \mathcal{F}_0, \gamma)$.

Definition 2: For each nonnegative integer ℓ and each function $f \in L_p(\Omega, \mathcal{F}_0, \gamma)$, let $\Gamma_\ell f \equiv \lim_{n\to\infty} \Lambda_n T^{n-\ell} f$.

Since the maps Γ_ℓ are limits of positive L_p-contractions they are also positive L_p-contractions.

3.3 The Isometry $\tilde{W}_p$ on Tail Functions. Any function of the form $\Gamma_\ell f$ is measurable with respect to the tail σ-algebra $\mathcal{H}_\infty$, which is defined as the intersection of all $\mathcal{H}_n$, $n \in \mathbb{Z}$. It is easy to show that $\tau(\mathcal{H}_\infty) = \mathcal{H}_\infty$.

We will relate the properties of W on $L_1(\Omega, \mathcal{H}_\infty, \gamma)$ to the limiting behavior of $T^n - T^{n+1}$ as $n \to \infty$.

Let $\tilde{\rho}$ denote the change of measure factor for τ when γ is restricted to $\mathcal{H}_\infty$, meaning that $\tilde{\rho}$ is nonnegative and $\mathcal{H}_\infty$-measurable, and such that

$$\int f \circ \tau d\gamma = \int \tilde{\rho} f d\gamma \tag{35}$$

for every nonnegative $\mathcal{H}_\infty$-measurable function f. Then

$$\tilde{\rho} = E_{\mathcal{H}_\infty} \rho. \tag{36}$$

Let

$$\tilde{W} f = \tilde{\rho}(f \circ \tau^{-1}), \ \tilde{W}_p g = \tilde{\rho}^{1/p}(g \circ \tau^{-1}), \tag{37}$$

so that W_p gives an isometry from $L_p(\Omega, \mathcal{H}_\infty, \gamma)$ into itself.

LEMMA 3: *For any nonnegative integer ℓ and any function $f \in L_p(\Omega, \mathcal{F}_0, \gamma)$ we have $\Gamma_\ell T f = \tilde{W}_p \Gamma_\ell f$.*

Proof: Since the operators involved are all linear we may assume that $f \geq 0$. Then

$$E_{\mathcal{H}_0} W^n \left(\Lambda_n T^{n-\ell} T f\right)^p = \left(T^{n+1-\ell} f\right)^p \tag{38}$$

$$= E_{\mathcal{H}_0} W^{n+1} \left(\Lambda_{n+1} T^{n+1-\ell} f\right)^p \tag{39}$$

$$= E_{\mathcal{H}_0} W^n E_{\mathcal{H}_n} W \left(\Lambda_{n+1} T^{n+1-\ell} f\right)^p, \tag{40}$$

and we have

$$\left(\Lambda_n T^{n-\ell} T f\right)^p = E_{\mathcal{H}_n} W \left(\Lambda_{n+1} T^{n+1-\ell} f\right)^p \tag{41}$$

$$= (E_{\mathcal{H}_n} \rho) \left(\Lambda_{n+1} T^{n+1-\ell} f\right)^p \circ \tau^{-1}. \tag{42}$$

Letting $n \to \infty$ and recalling that τ^{-1} is nonsingular, the lemma follows. ∎

3.4 RELATING T AND $\tilde{W}_p$.

LEMMA 4: *If $\tilde{W}_p$ is equal to the identity operator on those functions in $L_p(\Omega, \mathcal{H}_\infty, \gamma)$ which are supported on $\{\Gamma_0 \mathbf{1} > 0\}$, then $T^n f - T^{n+1} f \to 0$ strongly as $n \to \infty$.*

Proof: Let $f \in L_p(\Omega, \mathcal{F}_0, \gamma)$. Then

$$\Lambda_n (T^n f - T^{n+1} f) \to \Gamma_0 f - \Gamma_0 T f = \Gamma_0 f - \tilde{W}_p \Gamma_0 f = 0. \tag{43}$$

Since each Λ_n is an isometry this proves the lemma. ∎

Definition 3: For any positive integer m, we will say that $\tilde{W}_p$ contains an exact circle of length m if there exists $f \in L_p^+(\Omega, \mathcal{H}_\infty, \gamma)$ with $f \neq 0$ such that $\{f > 0\} \subset \{\Gamma_0 \mathbf{1} > 0\}$,

$$\tilde{W}_p^i f \wedge \tilde{W}_p^j f = 0 \quad \text{for } 0 \leq i < j < m, \tag{44}$$

and

$$\tilde{W}_p^m f = f. \tag{45}$$

Similarly, for any positive integer m, we will say that $\tilde{W}_p$ contains an exact line segment of length m if there exists $f \in L_p^+(\Omega, \mathcal{H}_\infty, \gamma)$ with $f \neq 0$ such that $\{f > 0\} \subset \{\Gamma_0 \mathbf{1} > 0\}$ and condition (44) holds.

LEMMA 5: *Let $f \in L_p(\Omega, \mathcal{H}_\infty, \gamma)$, such that $\{|f| > 0\} \subset \{\Gamma_0 \mathbf{1} > 0\}$. There exists $f_n \in L_p(\Omega, \mathcal{F}_0, \gamma)$, $n = 0, 1, \ldots$, such that*

$$\Gamma_n f_n \to f \tag{46}$$

as $n \to \infty$, both almost surely and in L_p-norm.

Proof: By linearity we may assume that $f \geq 0$. Using a simple approximation argument, we may also assume that $f = g\Gamma_0\mathbf{1}$, where g is nonnegative, bounded and $\mathcal{H}_\infty$-measurable. Using linearity again, we may assume that $0 \leq g \leq 1$.

Let

$$g_n = E_{\mathcal{G}_n} g, \quad f_n = g_n \circ \tau^{-n}\, T^n \mathbf{1}. \tag{47}$$

By the Markov property (15), g_n is $\mathcal{F}_n$-measurable, so f_n is $\mathcal{F}_0$-measurable.

Let $g_1 = g$ and let $g_2 = 1 - g$. Let $g_{n,1} = g_n$ and let $g_{n,2} = 1 - g_n$. By the martingale theorem, $g_{n,j} \to g_j$ almost surely as $n \to \infty$. The sequences $(g_{n,j})_{n=1,2,\ldots}$, $j = 1, 2$, are uniformly bounded.

Let $f_{n,j} = g_{n,j} \circ \tau^{-n}\, T^n \mathbf{1}$, for $j = 1, 2$, so that $f_{n,1} = f_n$ and $f_{n,2} = T^n\mathbf{1} - f_{n,1}$. Let $f_j = g_j \Gamma_0 \mathbf{1}$, so that $f_1 = f$. Since $(\Lambda_n T^n \mathbf{1})^p \to (\Gamma_0 \mathbf{1})^p$ as $n \to \infty$, both almost surely and in L_1-norm, it is easy to see using (26) that

$$(\Lambda_n f_{n,j})^p = (g_{n,j} \Lambda_n T^n \mathbf{1})^p \to (g_j \Gamma_0 \mathbf{1})^p = f_j^p \tag{48}$$

as $n \to \infty$, both almost surely and in L_1-norm, for $j = 1, 2$.

Since taking conditional expectation preserves almost sure and L_1 convergence, it is also true that

$$E_{\mathcal{H}_\infty} (\Lambda_n f_{n,j})^p \to f_j^p \tag{49}$$

as $n \to \infty$, both almost surely and in L_1-norm, for $j = 1, 2$.

Since $\left((\Lambda_m T^{m-n} f_{n,j})^p, \mathcal{H}_m\right)_{m=n,n+1,\ldots}$ is a backwards submartingale which converges in L_1-norm to $(\Gamma_n f_{n,j})^p$, we have

$$(\Gamma_n f_{n,j})^p \leq E_{\mathcal{H}\infty} (\Lambda_n f_{n,j})^p. \tag{50}$$

Hence

$$(\Gamma_n f_{n,j})^p \vee f_j^p \to f_j^p \tag{51}$$

as $n \to \infty$, both almost surely and in L_1-norm, for $j = 1, 2$, and so of course

$$(\Gamma_n f_{n,j}) \vee f_j \to f_j \tag{52}$$

as $n \to \infty$, both almost surely and in L_p-norm, for $j = 1, 2$.

Hence

$$(\Gamma_n f_{n,1}) \vee f_1 - f_1 + (\Gamma_n f_{n,2}) \vee f_2 - f_2 \to 0 \tag{53}$$

as $n \to \infty$, both almost surely and in L_p-norm.

But we also know by linearity that

$$\Gamma_n f_{n,1} + \Gamma_n f_{n,2} = \Gamma_n T^n \mathbf{1} = \Gamma_0 \mathbf{1} = f_1 + f_2. \tag{54}$$

Thus we also have

$$(\Gamma_n f_{n,1}) \vee f_1 - \Gamma_n f_{n,1} + (\Gamma_n f_{n,2}) \vee f_2 - \Gamma_n f_{n,2} \to 0 \tag{55}$$

as $n \to \infty$, both almost surely and in L_p-norm. Adding, we finally conclude that

$$|\Gamma_n f_{n,1} - f_1| + |\Gamma_n f_{n,2} - f_2| \to 0 \tag{56}$$

as $n \to \infty$, both almost surely and in L_p-norm, and this proves the lemma. ∎

LEMMA 6: *For any positive integer m, if $\tilde{W}_p$ contains an exact circle of length m then T contains an asymptotic circle of length m, and if $\tilde{W}_p$ contains an exact line segment of length m then T contains an asymptotic line segment of length m.*

Proof: Suppose that $\tilde{W}_p$ contains an exact circle of length m. Let $f \in L_p^+(\Omega, \mathcal{H}_\infty, \gamma)$ with $f \neq 0$ such that $\{f > 0\} \subset \{\Gamma_0 \mathbf{1} > 0\}$ and such that (44) and (45) hold. We may assume that $\|f\|_p > 2$. Let $\varepsilon > 0$ be given. We may assume that $\varepsilon < 1$. By Lemma 5 we can choose some nonnegative integer ℓ and some function $f_\ell \in L_p(\Omega, \mathcal{F}_0, \gamma)$ such that $\|\Gamma_\ell f_\ell - f\|_p < \varepsilon$. Then in particular we have $\|\Gamma_\ell f_\ell\|_p > 1$. Since $\Lambda_n T^{n-\ell} f_\ell \to f$ in L_p-norm, and Λ_n is an isometry, we see that

$$\lim_{n\to\infty} \|T^n f_\ell\|_p > 1. \tag{57}$$

Also, by the definition of Γ_ℓ there exists $n_0 \geq \ell$ such that for all $n \geq n_0$ we have

$$\|\Lambda_n T^{n-\ell} T^i f_\ell - \Gamma_\ell T^i f_\ell\|_p < \varepsilon \tag{58}$$

for $i = 0, 1, \ldots, m$. We see from (44) that

$$\|\tilde{W}_p^i \Gamma_\ell f_\ell \wedge \tilde{W}_p^j \Gamma_\ell f_\ell\|_p < 2\varepsilon \quad \text{for } 0 \leq i < j < m, \tag{59}$$

and hence that

$$\|\Gamma_\ell T^i f_\ell \wedge \Gamma_\ell T^j f_\ell\|_p < 2\varepsilon \quad \text{for } 0 \le i < j < m. \tag{60}$$

Thus for all $n \ge n_0$ we have

$$\|\Lambda_n T^{n+i-\ell} f_\ell \wedge \Lambda_n T^{n+j-\ell} f_\ell\|_p < 4\varepsilon \quad \text{for } 0 \le i < j < m. \tag{61}$$

Using the fact that Λ_n is an isometry, we see that for all $k \ge n_0 - \ell$ we have

$$\|T^{k+i} f_\ell \wedge T^{k+j} f_\ell\|_p < 4\varepsilon \quad \text{for } 0 \le i < j < m. \tag{62}$$

Similarly, we find that for all $k \ge n_0 - \ell$ we have

$$\|T^{k+m} f_\ell - T^k f_\ell\|_p < 4\varepsilon. \tag{63}$$

Since $\varepsilon > 0$ is arbitrary, it follows that T contains an asymptotic circle of length m. The proof of the line segment case is the same as the proof of (62) just given, so the lemma is proved. ∎

3.1 Finishing the Proof of Theorem 3 in the Dilation Case. Since we have just shown (Lemma 6) that T will contain an asymptotic circle or line segment if $\tilde{W}_p$ contains an exact circle or exact line segment, respectively, and we have also shown (Lemma 4) that $T^n - T^{n+1} \to 0$ strongly if $\tilde{W}_p$ is the identity operator on those functions in $L_p(\Omega, \mathcal{H}_\infty, \gamma)$ which are supported on $\{\Gamma_0 \mathbf{1} > 0\}$, the proof of Theorem 3 in the dilation case will be complete as soon as we establish the next lemma.

Lemma 7: *Consider the following two statements.*

(a) $\tilde{W}_p$ *contains an exact circle of length* $m \ge 2$.

(b) $\tilde{W}_p$ *contains exact line segments of length* m *for every positive integer* m.

Suppose neither of these statements (a) *and* (b) *is true. Then* $\tilde{W}_p$ *must be equal to the identity operator on those functions in* $L_p(\Omega, \mathcal{H}_\infty, \gamma)$ *which are supported on* $\{\Gamma_0 \mathbf{1} > 0\}$.

Proof: We will prove the lemma by assuming that $\tilde{W}_p$ is not the identity operator on those functions in $L_p(\Omega, \mathcal{H}_\infty, \gamma)$ which are supported on $\{\Gamma_0 \mathbf{1} > 0\}$, and then showing that at least one of conditions (a) and (b) must hold. The argument is essentially a repetition of the old proof from ergodic theory of the existence of a Rohlin tower for an aperiodic point transformation.

As a convenient notation, we will write

$$\Upsilon = \{\Gamma_0 \mathbf{1} > 0\}. \tag{64}$$

We note that Υ is only defined up to a set of γ-measure zero, and any properties that we prove for Υ must be true for any version of this set.

Since $\tilde{W}_p\Gamma_0\mathbf{1} = \Gamma_0 T\mathbf{1}$, it is easy to show that

$$\{\tilde{W}_p\Gamma_0\mathbf{1} > 0\} \subset \{\Gamma_0\mathbf{1} > 0\}. \tag{65}$$

We have $\tilde{W}_p\Gamma_0\mathbf{1} = \tilde{\rho}^{1/p}(\Gamma_0\mathbf{1})\circ\tau^{-1}$, and hence $\{\tilde{W}_p\Gamma_0\mathbf{1} > 0\} = \{\tilde{\rho} > 0\} \cap \{\tau(\Upsilon)\}$. Since $\tilde{\rho} > 0$ almost everywhere, we have $\gamma(\tau(\Upsilon) - \Upsilon) = 0$. Removing a set of measure 0 from Υ we may assume that $\tau(\Upsilon) \subset \Upsilon$.

Suppose that for every $A \in \mathcal{H}_\infty$ with $A \subset \Upsilon$ we have $\gamma(A\Delta\tau(A)) = 0$, and hence that $\gamma(\tau^{-1}(A)\Delta A) = 0$. Then

$$\gamma(A) = \gamma(\tau^{-1}(A)) = \int_A \tilde{\rho}\, d\gamma \tag{66}$$

for all $A \in \mathcal{H}_\infty$ with $A \subset \Upsilon$, so that $\tilde{\rho} \equiv 1$ almost everywhere on Υ. Hence $\tilde{W}_p$ is equal to the identity on functions of the form $\mathbf{1}_A$, $A \in \mathcal{H}_\infty$ with $A \subset \Upsilon$. By the usual argument of approximating nonnegative functions by simple functions, we see that $\tilde{W}_p$ is equal to the identity on those functions in $L_p(\Omega, \mathcal{H}_\infty, \gamma)$ which are supported on Υ, contradiction. Therefore we have shown that for some set $A \in \mathcal{H}_\infty$ with $A \subset \Upsilon$ we have $\gamma(A\Delta\tau(A)) > 0$.

Since $\gamma(A\Delta\tau(A)) > 0$ then either $\gamma(A - \tau(A)) > 0$ or $\gamma(\tau(A) - A) > 0$. In the first case let $B = A - \tau(A)$ and in the second case let $B = A - \tau^{-1}(A)$. In either case we have $\gamma(B) > 0$ and $\gamma(B \cap \tau(B)) = 0$, and we see from the definition of $\tilde{W}_p$ that $\tilde{W}_p$ has an exact line segment of length 2.

If statement (b) does not hold then let n be the largest integer such that for some $B \in \mathcal{H}_\infty$ with $B \subset \Upsilon$ we have $\gamma(B) > 0$ and $\gamma(\tau^i(B) \cap \tau^j(B)) = 0$ for $0 \le i < j < n$.

If there is a $\mathcal{K}_\infty$-measurable subset C of B such that $\gamma(C\Delta\tau^n(C)) > 0$ then either $\gamma(C - \tau^n(C)) > 0$ or $\gamma(\tau^n(C) - C) > 0$. In the first case let $D = C - \tau^n(C)$ and in the second case let $D = C - \tau^{-n}(C)$. In either case we have $\gamma(D) > 0$ and $\gamma(D \cap \tau^n(D)) = 0$. It follows easily that this contradicts the maximality of n. Consequently, we have $\gamma(C\Delta\tau^n(C)) = 0$ for every $\mathcal{K}_\infty$-measurable subset C of B. Thus $\tilde{\rho}_n \equiv 1$ holds almost everywhere on B, where $\tilde{\rho}_n$ is the change of measure factor for τ^n. Since $\tilde{W}_p^n f = \tilde{\rho}_n f \circ \tau^{-n}$, $\tilde{W}_p^n \mathbf{1}_B = \mathbf{1}_B$. By considering $f = \mathbf{1}_B$ we see that $\tilde{W}_p$ contains an exact circle of length n, so statement (a) holds for $m = n$, and the lemma is proved. ∎

This completes that proof of Theorem 3 in the dilation case.

4. Reduction to the dilation case

4.1 PRELIMINARIES. It is an easy exercise to show that the proof of Theorem 3 can be reduced to the case in which μ is σ-finite, and thence to the case in which μ is a probability measure. The reduction to the case of a probability measure can always be carried out after any other reductions, so we will not need to normalize our measure space while discussing other reductions. Easy standard arguments allow us to assume without loss of generality that X is a compact metric space and $\mathcal{F}$ is the σ-algebra of Borel subsets. We will omit these details. However, a little more work is needed to show that without loss of generality we may also assume that the operator T has full range and domain, or, more briefly, that T is a full operator. By full range we mean that $T\mathbf{1} > 0$ holds μ-almost everywhere, and by full domain we mean that T has trivial nullspace. We will consider the nullspace of T first.

4.2 FULL DOMAIN.

4.2.1 The supporting set of a cone. In the context of a function space, a cone is a set of functions which is closed under the operations of addition and multiplication by nonnegative real numbers.

Let C be a cone in $L_p^+(X, \mathcal{F}, \mu)$ which is also closed in norm. Since we are now assuming that μ is σ-finite, let ν be a bounded measure which is equivalent to μ. Define

$$\alpha(C) = \sup\{\nu(\{f > 0\}) : \ f \in C\}. \tag{67}$$

Let $f_n \in C$ be such that

$$\mu(\{f_n > 0\}) \to \alpha(C). \tag{68}$$

Let D be the union of the sets $\{f_n > 0\}$. It is easy to see that there is a function $f \in C$ such that $D = \{f > 0\}$ and also that if $g \in C$ then

$$\mu(\{g > 0\} - D) = 0. \tag{69}$$

Thus D is the unique maximal supporting set of a function in C. We will call D the supporting set of the cone C. Every function g in the cone is supported on D.

4.2.2 Aymptotically null functions. We will say that a function f in $L_p^+(X, \mathcal{F}, \mu)$ is asymptotically T-null if

$$\lim_{n\to\infty} \|T^n f\|_p = 0. \tag{70}$$

It is obvious from the definition that the set of asymptotically T-null functions is closed under T. Since T is a contraction, the set of asymptotically T-null functions is clearly a closed cone in $L_p^+(X,\mathcal{F},\mu)$.

LEMMA 8: *Let f be asymptotically T-null. Let $B = \{x\colon f(x) > 0\}$. Then for any function $g \in L_p^+(X,\mathcal{F},\mu)$, the function $\mathbf{1}_B g$ is also asymptotically T-null.*

Proof: Since the asymptotically T-null functions form a closed set, it is enough to consider g bounded, and hence to consider g bounded by 1. Let $B_n = \{x\colon f(x) > 1/n\}$. Since

$$\mathbf{1}_{B_n} \nearrow \mathbf{1}_B, \tag{71}$$

we have $\mathbf{1}_{B_n} g \to \mathbf{1}_B g$ in L_p-norm, so it is enough to show that $\mathbf{1}_{B_n} g$ is asymptotically T-null. Since

$$\mathbf{1}_{B_n} g \leq nf \tag{72}$$

this is obvious, and the lemma is proved. ∎

Let $\tilde{X}$ be the supporting set of the cone of asymptotically T-null functions. By what has been shown, a function $g \in L_p^+(X,\mathcal{F},\mu)$ is asymptotically T-null if and only if g is supported on $\tilde{X}$. Let $\hat{X} = X - \tilde{X}$ and let $S = \mathbf{1}_{\hat{X}} T$. Clearly, if g is supported on $\tilde{X}$ then so is Tg, and so $Sg = 0$.

LEMMA 9: *Let $f \in L_p(X,\mathcal{F},\mu)$. Then for any $n = 0, 1, \ldots$, $T^n f = S^n f + g_n$ where g_n is supported on $\tilde{X}$. Furthermore, $g_n \to 0$ as $n \to \infty$.*

Proof: The first statement is an easy induction using the fact that $Th - Sh$ is supported on $\tilde{X}$ for any h.

Let m be a nonnegative integer. For any nonnegative integer n we have $\|T^{n+m} f\|_p \leq \|T^n S^m f\|_p + \|T^n g_m\|_p$. Since g_m is asymptotically T-null we see that

$$\lim_{n\to\infty} \|T^n f\|_p \leq \lim_{m\to\infty} \|S^m f\|_p. \tag{73}$$

The reverse inequality is clear, so the two limits are equal. But $\|T^n f\|_p^p = \|S^n f\|_p^p + \|g_n\|_p^p$, and the lemma follows. ∎

Using Lemma 9 it is easy to show that in proving Theorem 3 we may assume without loss of generality that $\mu(\tilde{X}) = 0$. From now on we do assume this, so that the only asymptotically T-null function is the zero function. In particular, T has only a trivial nullspace, i.e. T has full domain.

4.3 FULL RANGE. We now consider the range of T. Let $B = X - \{T\mathbf{1} > 0\}$, and suppose that $\mu(B) > 0$. Let B' be a copy of B. We consider a new space $X' \equiv X \cup B'$, with measure μ' defined equal to μ on measurable subsets of X and with $\mu'(A') \equiv \mu(A)$ for any measurable subset A' of B', where A denotes the corresponding subset of B. We define a positive L^p-contraction T' on $L_p(\mu')$ as follows. If f is supported on X, we define $T'f$ to be equal to Tf on X and equal to zero on B'. If f' is supported on B', we let $T'f' = (1/2)f' + (1/2)f$, where f is the function supported on B which corresponds to f'. It is immediate that T' is a positive L^p contraction, and also that T' has full range. Since T can be regarded as the restriction of T' to an invariant subspace of its domain, it is obvious that $T^n - T^{n+1} \to 0$ if $(T')^n - (T')^{n+1} \to 0$. Suppose that T' contains an asymptotic circle of length m. Then there is some positive L_p-function f such that for large n the finite sequence $(T')^{n+j}f$, $j = 0, 1, \ldots, m$ satisfies the conditions described in Definition 1. For any $\varepsilon > 0$, for sufficiently large n we clearly have $\|(T')^n f - h_n\|_p < \varepsilon$, where h_n is equal to $(T')^n f$ on X and $h_n \equiv 0$ on B'. It follows that T also contains an asymptotic circle of length m. Similarly, if T' contains an asymptotic line segment of length m then T also contains an asymptotic line segment of length m. Thus Theorem 3 for T' immediately implies the result for T. We also note that T' inherits properties of T that we have previously assumed, for example, T' has no asymptotically null functions if T has no asymptotically null functions. Thus without loss of generality we may assume in the proof of Theorem 3 that, in addition to all previous assumptions, the operator T has full range.

4.2 THE DILATION REPRESENTATION FOR A FULL OPERATOR. The reductions made so far show that in proving Theorem 3 it is enough to consider the case of a full positive L_p-contraction on the L_p-space associated with a probability measure on the Borel sets of a compact metric space. We will make one further reduction, namely to restrict our attention to the case of measure spaces which have been "enriched" by taking the Cartesian product with a copy of the unit interval, equipped with the usual Lebesgue probability measure. This extra factor in the measure space is completely independent of the action of the operator T under consideration, and in an obvious way we may consider T to be defined on the L_p-space of the new product measure. It is easy to show that if we can prove Theorem 3 for a new operator then it will hold for the original T. We will show that the new operator T on the enriched product space is equivalent, via an L_p-isometry, to an operator which satisfies all the assumptions of the dilation case. This observation will complete the proof of Theorem 3.

The representation result we need is given, in various forms, in each of [4], [6] and [3]. We will follow [3] here. Actually [3] obtains a simultaneous representation for countable family of pairs of operators, but we only need the result for a single operator, and we will extract this part.

For any full operator (also defined in Section 2.4 of [3]) on the L_p-space associated with a probability measure on the Borel sets of a compact metric space, a representation using conditional expectation and an L_p-isometry is obtained in Theorem 3.14 of [3]. The existence of the isometry in Theorem 3.14 is based on a measure algebra isomorphism K constructed in Lemma 3.9 of [3]. Because we have enriched our measure space, the map K can be chosen (using the notation of Lemma 3.9) to be a measure-preserving isomorphism from $(\mathcal{F} \times \mathcal{G}, \xi)$ onto the full product measure space on $X \times [0,1]$, i.e. we can take the σ-algebra $\mathcal{R}$ of that lemma to be the full product σ-algebra on $X \times [0,1]$. This form of Lemma 3.9 implies that in Theorem 3.14 the isomorphism M can be chosen to be a measure algebra isomorphism between the full product σ-algebras on $X \times [0,1]$ and $Y \times [0,1]$. In turn, this form of Theorem 3.14 means that the measure algebra isomorphism M constructed in Section 5.2 of [3] can be taken to be a measure algebra isomorphism from the full σ-algebra Σ to itself. The space (Ω, Σ, γ) of the present paper is the space $(\Omega, \Sigma, \vartheta)$ of Section 5.2 of [3], while the σ-algebra $\mathcal{F} = \mathcal{F}_0$ of the present paper is the σ-algebra Σ_0 of [3] generated by the single coordinate ω_0. The mapping τ of the present paper is defined so that the measure algebra isomorphism M in Section 5.2 of [3] satisfies $M(A) = \tau(A)$, i.e. M is simply τ considered as a set map.

The only other properties of the dilation case which are not explicitly discussed in [3] are the Markov properties (14) and (15). As mentioned earlier, it is an easy standard fact in the theory of Markov processes that these two properties are equivalent, so we need only prove one of them. Since the mapping τ of the present paper is not assumed to be the coordinate shift on infinite product space, it seems better to consider the case $n = 0$ separately. In the $n = 0$ case, we see that in terms of the infinite product space constructed in Section 5.1 of [3], our σ-algebra $\mathcal{G}_0$ is contained in the σ-algebra generated by the coordinates ω_n, $n \geq 0$, whereas our σ-algebra $\mathcal{H}_0$ is contained in the σ-algebra generated by the coordinates ω_n, $n \leq 0$. Since the factors in the infinite product space are independent, it is then trivial to prove the Markov properties for $n = 0$. For general $n \geq 0$, we make one more observation, namely that the change-of-measure factor ρ associated with our point transformation τ is $\mathcal{G}_0$-measurable. The following general lemma then gives the result we need.

LEMMA 10 (Markov Properties): *In the setting of Section 2, assume that (15) holds for $n = 0$, and that the change-of-measure factor is $\mathcal{G}_0$-measurable. Then (15) holds for all $n \geq 0$.*

Proof: We will write $W^n f = \rho_n(f \circ \tau^{-n})$, where $\rho_n = W^n \mathbf{1}$. Since ρ is $\mathcal{G}_0$-measurable, we see that ρ_n is also $\mathcal{G}_0$-measurable, noting that

$$\rho_n = \rho(\rho \circ \tau^{-1}) \cdots (\rho \circ \tau^{-n+1}). \tag{74}$$

Let h be bounded and $\mathcal{H}_n$-measurable. It is enough to show that $E_{\mathcal{G}_n} h$ is $\mathcal{F}_n$-measurable. Note that $h \circ \tau^{-n}$ is $\mathcal{H}_0$-measurable. Since

$$W^n E_{\mathcal{G}_n} h = \rho_n \left(E_{\mathcal{G}_n} h\right) \circ \tau^{-n}, \tag{75}$$

and Lemma 1 does not depend on (14) or (15), we have

$$\begin{aligned} W^n E_{\mathcal{G}_n} h &= E_{\mathcal{G}_0} W^n E_{\mathcal{G}_n} h && (76)\\ &= E_{\mathcal{G}_0} W^n h && (77)\\ &= E_{\mathcal{G}_0} \rho_n (h \circ \tau^{-n}) && (78)\\ &= \rho_n E_{\mathcal{G}_0} (h \circ \tau^{-n}) && (79)\\ &= \rho_n E_{\mathcal{F}_0} (h \circ \tau^{-n}) && (80)\\ &= W^n \left(E_{\mathcal{F}_0}(h \circ \tau^{-n})\right) \circ \tau^n. && (81) \end{aligned}$$

Since $(E_{\mathcal{F}_0}(h \circ \tau^{-n})) \circ \tau^n$ is $\mathcal{F}_n$-measurable and W^n is one-to-one, the lemma is proved. ∎

This finishes the reduction to the dilation case, and finishes the proof of Theorem 3.

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